Controllability of a string submitted to unilateral constraints

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Introduction

Let $T > 0$. We consider the nonlinear control problem:

$$\begin{cases} y'' - y_{xx} = 0, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = u(t), & t \in (0,T), \\ \begin{array}{l} y(t,1) \geq \psi(t), \ y_x(t,1) \geq 0, \\ (y(t,1) - \psi(t))y_x(t,1) = 0, \end{array} & t \in (0,T), \\ y(0,x) = y^0(x), \quad y'(0,x) = y^1(x), & x \in (0,1) \end{cases} \tag{1}$$

- $u \in L^2(0,T)$ will be a control function;
- $\psi \in H^1(0,T)$ is a given obstacle;
- $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$.

Introduction

- Problem: Given $T > 0$ and $\left(y^0, y^1\right), \left(z^0, z^1\right) \in L^2(0,1) \times H^{-1}(0,1)$, does there exist $u \in L^2(0, T)$ such that $y(T) \equiv z^0, y'(T) \equiv z^1$ on $(0,1)$?

Bibliography

- Well-posedness in higher space dimensions:

1. G. Lebeau and M. Schatzman, *A wave problem in a half space with a unilateral constraint at the boundary*, J. Differential Equations, 53 (1984), 309-361.
2. J.U. Kim, *A boundary thin obstacle problem for a wave equation*, Commun. in Partial Differential Equation, 14 (1989), 1011-1026.

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These works just deal with well-posedness (no control) of:

$$\begin{cases} y'' - \Delta y = 0, & (0,T)\times\Omega \\ y \geq 0,\ \frac{\partial y}{\partial n} \geq 0 & \\ & (0,T)\times\partial\Omega \\ y.\ \frac{\partial y}{\partial n} = 0 & \\ y^0, y^1 & \end{cases}$$

Lebeau-Schatzman for particular domains and Kim for smooth domains.

Bibliography

- Control and stabilization:

Bibliography

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1. J.E. Rivera and H.P. Oquendo, *Exponential decay for a contact problem with local damping*, Funkcialaj Ekvacioj, 42 (1999), 371-387. Exponential decay for

$$\begin{cases} y'' - y_{xx} = -a(x)y', & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = 0, & t \in (0,T), \\ \begin{array}{l} y(t,1) \geq 0, \ y_x(t,1) \geq 0, \\ y(t,1)y_x(t,1) = 0, \end{array} & t \in (0,T), \\ y(0,x) = y^0(x), \quad y'(0,x) = y^1(x), & x \in (0,1) \end{cases}$$

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y(0,x) = y^0(x), \quad y'(0,x) = y^1(x), & x \in (0,1)
\end{cases}
$$

2. D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions*, SIAM Review, 20 (1978), 639-739.
   The characteristics method for the controllability of hyperbolic one dimensional systems...

Main result

We will sketch the proof of the following null-controllability result:

Theorem

Let $T > 2$ *and* $\psi \in H^1(0,T)$ *with the property that* $\psi(T) < 0$. *For any* $(y^0, y^1) \in H^1(0,1) \times \ L^2(0,1)$ *with*

$$\psi(0) \leq y^0(1),$$

there exists a control function $u \in H^1(0,T)$ *such that (1) admits a unique solution*

$$y \in C\left([0,T], H^1(0,1)\right) \cap C^1\left([0,T], L^2(0,1)\right)$$

satisfying

$$y(T) = y'(T) = 0, \ \text{in } (0,1).$$

Outline of the proof

- **First step**: consider the linear control problem:

$$\begin{cases} \varphi'' - \varphi_{xx} = 0, & (t,x) \in (0,T) \times (0,1), \\ \varphi(t,0) = u(t), & t \in (0,T), \\ \varphi(t,1) = f(t), & \\ \varphi(0,x) = y^0(x), & x \in (0,1) \\ \varphi'(0,x) = y^1(x), & \end{cases} \tag{2}$$

and compute the set $U = \{(u,f) : \varphi(T) = \varphi'(T) = 0 \text{ on } (0,1)\}$.
Main tool: the characteristics method.

Outline of the proof

- **Second step**: consider and study the Dirichlet-Neumann map:

$$A\left(y^0, y^1, u, f\right) = \varphi_x(\cdot, 1)$$

Main tool: an idea from Lebeau-Schatzman

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Main tool: an idea from Lebeau-Schatzman

- **Third step**: Study the equivalent problem:

$$\begin{array}{c} f \geq \psi \\ A\left(y^0, y^1, u, f\right) \geq 0 \\ (f-\psi)\, A\left(y^0, y^1, u, f\right) = 0 \end{array} \quad , \quad (0, T). \tag{3}$$

Main tool: Differential inequalities or penalty method.

First step: well-posedness

Lemma

If $\left(y^0, y^1\right) \in \left(H^1 \times L^2\right)(0,1)$, $(u, f) \in H^1(0, T)^2$ *and verify*

$$u(0) = y^0(0)\,,\ \ f(0) = y^0(1)\,,$$

then there exists a unique solution
$\varphi \in C\left(0, T; H^1(0,1)\right) \cap C^1\left(0, T; L^2(0,1)\right)$ *of Problem* (2) *such that*

$$\left\|\left(\varphi, \varphi'\right)(t)\right\|_{H^1 \times L^2(0,1)} \leq C\left(\left\|y^0, y^1\right\|_{H^1 \times L^2(0,1)} + \left\|(u, f)\right\|_{H^1(0,T)^2}\right)$$

First step: the set U

Lemma

With the assumptions of the previous lemma, let $T \in (2,3)$ *. Then the solution* φ *satisfies* $\varphi(T) = \varphi'(T) = 0$ *if, and only if*

$$\begin{cases} u'(t) = f'(t+1) + \frac{1}{2}q^0(t), & T-2 < t < 1 \\ u'(t) = f'(t+1) + f'(t-1) - \frac{1}{2}p^0(2-t) & 1 < t < T-1 \\ u'(t) = f'(t-1) - \frac{1}{2}p^0(2-t) & T-1 < t < 2 \\ u'(t) + u'(t-2) = f'(t-1) + \frac{1}{2}q^0(t-2), & 2 < t < T \end{cases}$$

where $p = \varphi' - \varphi_x$ *and* $q = \varphi' + \varphi_x$ *.*

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where $p = \varphi' - \varphi_x$ *and* $q = \varphi' + \varphi_x$.

Hint: There is no condition on $(0, T-2)$ for (u, f) and these relations define the set U.

Second step: the control Dirichlet-to-Neumann map

Lemma

Under the assumptions of the two previous lemmas, we have:

$$A\left(y^0, y^1, u, f\right)(t) = \begin{cases} f'(t) - p^0\left(1-t\right), & 0 < t < 1 \\ f'(t) - 2u'\left(t-1\right) + q^0\left(t-1\right), & 1 < t < T-1 \\ -f'(t), & T-1 < t < T \end{cases}$$

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- Note that the expression of $A\left(y^0, y^1, u, f\right)$ only involves the part of u defined on $\left(0, T-2\right)$, i. e. *the free part of u*.

Third step: an equivalent problem

Problem (3) now writes: with

$$v(t) = \begin{cases} p^0(1-t), & 0 < t < 1 \\ 2u'(t-1) - q^0(t-1), & 1 < t < T-1 \end{cases}$$

- **On** $(0, T-1)$:

$$\begin{cases} f - \psi \geq 0, \\ f' - v \geq 0, \\ (f-\psi)(f'-v) = 0, \\ f(0) = y^0(1), \end{cases}$$

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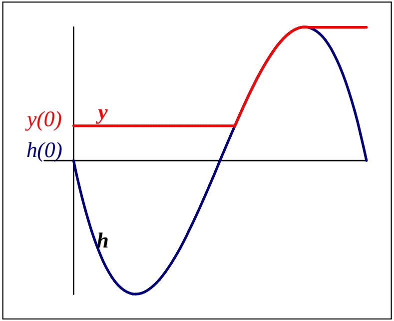
- **On** $(T-1, T)$:

$$\begin{cases} f - \psi \geq 0, \\ f'(t) \leq 0, \\ (f-\psi) f' = 0, \\ f(T) = 0. \end{cases}$$

A differential inequality

Lemma. *Let* $h \in H^1(0,T)$ *and* $\theta^0 \geq h(0)$. *Then the function* $\theta(t) = \max\left(\theta^0, \sup_{0\leq s\leq t} h(s)\right)$ *is the unique solution in* $H^1(0,T)$ *of*

$$\begin{cases} \theta - h \geq 0, \\ \theta' \geq 0, \\ (\theta - h)\,\theta' = 0, \\ \theta(0) = \theta^0. \end{cases}$$



- Bénilan-Pierre ('79) treats much more intricate situations

Proof: using differential inequalities

With this lemma, we get:

- **On** $(0, T-1)$:

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$$\Rightarrow \; f(t) = \int_0^t v + \max\left(y^0(1), \sup_{0\leq s\leq t} \left(\psi(s) - \int_0^s v\right)\right)$$

- **On** $(T-1, T)$:

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$$\begin{cases} f-\psi \geq 0, \\ f'(t) \leq 0, \\ (f-\psi) f' = 0, \\ f(T) = 0. \end{cases} \Rightarrow f(t) = \left[\sup_{0\leq s\leq t} \psi(s)\right]^+$$

Proof: using differential inequalities

- To end the proof with this method, it remains to prove that there exists u such that $f \in H^1(0, T)$.

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$$\lim_{t \to (T-1)^-} f(t) = \lim_{t \to (T-1)^+} f(t)$$

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- At this level, the fact that u is free on $(0, T-2)$ is used and this last condition is a control problem!

Comments

- An alternative proof of this result is to consider the penalized problem: $\varepsilon > 0$

$$\begin{cases} y_\varepsilon'' - y_{\varepsilon,xx} = 0, & (0,T)\times(0,1), \\ y_\varepsilon(t,0) = u(t), & (0,T), \\ y_{x,\varepsilon}(t,1) = \varepsilon^{-1}\left[y_\varepsilon(t,1) - \psi(t)\right]^- & (0,T), \\ y_\varepsilon(0,x) = y^0(x), \quad y_\varepsilon'(0,x) = y^1(x), & (0,1) \end{cases}$$

and use the control Dirichlet-to-Neumann map to reduce this problem to a *differential equation* on $(0,T)$ at $x=1$.

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  and use the control Dirichlet-to-Neumann map to reduce this problem to a *differential equation* on $(0,T)$ at $x=1$.
- If $T=2$, the two methods of proof do not work without restrictions on the initial data.

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- For details: FAK, S. Micu, A. Münch, *Controllability of a string submitted to unilateral constraint*, Annales de l'Institut Henri Poincaré - Analyse non linéaire (2010).

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- The same problem is open in higher space dimensions.

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- What is about the control parabolic problem:

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- and many others...

Thank you for your attention!